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String gradient weighted moving finite elements

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SUMMARY

Moving finite element methods are well established for solution of systems of partial differential equations which contain regions where the solution is rapidly varying but moving. The string or second gradient weighted moving finite element method (SGWMFE) uses a piecewise linear discretization of a single evolving manifold to approximate the solution of the PDEs.

In the case of one space dimension, x, and two dependent variables, $u(x,t)$ and $v(x,t)$, the solution is calculated from the normal motion of a single manifold $[x(\tau,t), u(\tau,t), v(\tau,t)]$, where τ is a parameter along the maniflold, or a 'string' embedded in $[x,u,v]$ space. This method can be extended to multiple dimensions and an arbitrary number of dependent variables in which case the 'string' parameterization analogy is replaced by a multi-variable parameterization.

In this paper, we outline the application of SGWMFE for solution of the shallow water equations in one and two space dimensions. We describe the results of a number of numerical experiments, including varying the initial distribution of nodes. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: moving finite elements; partial differential equations; r-adaptive mesh; moving mesh; adaptive grids

1. INTRODUCTION

The gradient weighted finite element method (GWMFE) was developed in References [1, 2] for one- and two-dimensional problems. A second GWMFE method for systems of PDEs was proposed by Reference [3] and its implementation for one-dimensional problems outlined in [4]. In this paper, we give brief details of the extension from one space dimension to two space dimensions using the shallow water equations as a model system of PDEs but do not discuss implementation issues. In Reference [4], the implementation of the SGWMFE method

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was aided for a system of PDEs in one dimension by use of a projection matrix. We describe the projection matrix for one dimension and its extension to two space dimensions.

As for other GWMFE solutions of the inviscid shallow water equations [5], we use added diffusion in the PDEs, with a small diffusion coefficient so as to complete the system of equations and to provide a classical solution. The addition of diffusion prevents the occurrence of infinite gradients and multivalued foldover. Thus, we compute with diffusion as small as we can handle (numerical difficulties arise because of the resulting exceedingly thin shocks) in order to approximate a desired 'zero diffusion limit' weak solution.

2. STRING GRADIENT WEIGHTED MOVING FINITE ELEMENTS

Consider, in one space dimension, the example system of PDEs

$$u_t = L_1(u,v), \quad v_t = L_2(u,v) \tag{1}$$

for the two unknown functions $u(x,t)$ and $v(x,t)$ on an interval. Here, L_1 and L_2 are first-or second-order nonlinear differential operators. The solution graphs for system (1) may be treated as a single evolving one-dimensional mainfold (a 'string') immersed in three dimensions, that is as $(x,u(x,t),v(x,t))$. Under reparamerterization with a moving variable $x(\tau,t)$, the string becomes an evolving parameterized manifold

$$\mathbf{u}(\tau,t) = (x(\tau,t),u(\tau,t),v(\tau,t)) \tag{2}$$

At each parameterized point on the evolving manifold, we divide the motion vector $\dot{\mathbf{u}} = (\dot{x},\dot{u},\dot{v})$ into its tangential, $[\dot{\mathbf{u}}]_{\mathrm{T}}$, and its normal, $[\dot{\mathbf{u}}]_{\mathrm{N}}$, parts. The original system (1) was the equation for the 'vertical' motion $(0,u_t,v_t)$ of the manifold $(0,L_1(u,v),L_2(u,v))$. Since $(x,u(x,t),v(x,t))$ is one parameterization of the solution manifold, we automatically have the same normal motion

$$\begin{pmatrix} \dot{x} \\ \dot{u} \\ \dot{v} \end{pmatrix}_{\mathrm{N}} = \begin{pmatrix} 0 \\ u_t \\ v_t \end{pmatrix}_{\mathrm{N}} = \begin{pmatrix} 0 \\ L_1 \\ L_2 \end{pmatrix}_{\mathrm{N}} \tag{3}$$

for any other parameterization. The equation for the normal motion is thus a system of three PDEs for the three unknown functions $x(\tau,t),u(\tau,t),v(\tau,t)$ (a degenerate system since the tangential component of the motion is left completely free).

The implementation was simplified in Reference [4] by using the projection matrix P, which projected any given vector, $\mathbf{F}$, into its normal part, $[\mathbf{F}]_{\mathrm{N}}$ at a given point on a 'string', $(x,u(x),v(x))$. In one dimension, the tangential projection is given by

$$[\mathbf{F}]_{\mathrm{T}} = D^{-1}\mathbf{X}\mathbf{X}^{\mathrm{T}}\mathbf{F} \tag{4}$$

where $\mathbf{X} = (1,u_x,v_x)^{\mathrm{T}}$ is a tangent vector to the manifold at this point and $D = \mathbf{X}^{\mathrm{T}}\mathbf{X}$. Hence,

$$[\mathbf{F}]_{\mathrm{N}} = \mathbf{F} - [\mathbf{F}]_{\mathrm{T}} = (\mathbf{I} - D^{-1}\mathbf{X}\mathbf{X}^{\mathrm{T}})\mathbf{F} = P\mathbf{F} \tag{5}$$

where

$$P = \mathbf{I} - D^{-1}\mathbf{X}\mathbf{X}^{\mathrm{T}} = \frac{1}{1+u_x^2+v_x^2}\begin{pmatrix} (u_x^2+v_x^2) & -u_x & -v_x \\ -u_x & (1+v_x^2) & -u_x v_x \\ -v_x & -v_x u_x & (1+u_x^2) \end{pmatrix} \tag{6}$$

is constant on each element when using piecewise linear discretization.

Following Reference [1], Equation (3) is treated geometrically-mechanically, as a balance of viscous drag forces and applied forces per unit arc length of the string, all in the normal direction. The force balance Equation, (3), is discretized by letting the SGWMFE approximant be an evolving, piecewise linear manifold with its three-dimensional nodal positions $\mathbf{u}_j = (x_j(t), u_j(t), v_j(t))$ as unknowns. The distributed normal forces are concentrated onto the nodes to give a balance of three-dimensional forces

$$\int \begin{pmatrix} \dot{x} \\ \dot{u} \\ \dot{v} \end{pmatrix}_{\mathrm{N}} \alpha^j \, \mathrm{d}s = \int \begin{pmatrix} 0 \\ L_1 \\ L_2 \end{pmatrix}_{\mathrm{N}} \alpha^j \, \mathrm{d}s \tag{7}$$

at each node, where s is the arc length along the manifold, $\mathrm{d}s = \sqrt{1+u_x^2+v_x^2}\ \mathrm{d}x = \sqrt{D}\,\mathrm{d}x$ and α^j is the usual piecewise linear 'hat' function centred on the jth node.

Consider now the case where the governing equations, (1), are dependent on two space variables. In this situation, the solution manifold is described by

$$\mathbf{u}(\tau,t) = (x(\tau,t), y(\tau,t), u(\tau,t), v(\tau,t)) \tag{8}$$

where $\tau = (\tau_1, \tau_2)$ is a two-dimensional parameter representing surface parameterization of the manifold.

In order to calculate the normal motion of the manifold, define two linearly independent tangent vectors to the manifold, $\mathbf{X} = (1, 0, u_x, v_x)^{\mathrm{T}}$ and $\mathbf{Y} = (0, 1, u_y, v_y)^{\mathrm{T}}$. Then, the normal component of any vector $\mathbf{F}$ can be written as $[\mathbf{F}]_{\mathrm{N}} = P\mathbf{F}$ where the projection matrix P is given by

$$P = \mathbf{I} - D^{-1}\{(\mathbf{Y}^{\mathrm{T}}\mathbf{Y})(\mathbf{X}\mathbf{X}^{\mathrm{T}}) - (\mathbf{X}^{\mathrm{T}}\mathbf{Y})(\mathbf{X}\mathbf{Y}^{\mathrm{T}} + \mathbf{Y}\mathbf{X}^{\mathrm{T}}) + (\mathbf{X}^{\mathrm{T}}\mathbf{X})(\mathbf{Y}\mathbf{Y}^{\mathrm{T}})\} \tag{9}$$

where $D = (\mathbf{X}^{\mathrm{T}}\mathbf{X})(\mathbf{Y}^{\mathrm{T}}\mathbf{Y}) - (\mathbf{X}^{\mathrm{T}}\mathbf{Y})^2$.

The equations of normal motion are discretized using piecewise linear representation on triangular elements in which case P is again constant on each element. As in one dimension, the mechanical interpretation results in the concentration of forces onto nodes by

$$\int \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{u} \\ \dot{v} \end{pmatrix}_{\mathrm{N}} \alpha^j \, \mathrm{d}S = \int \begin{pmatrix} 0 \\ 0 \\ L_1 \\ L_2 \end{pmatrix}_{\mathrm{N}} \alpha^j \, \mathrm{d}S \tag{10}$$

where $\mathrm{d}S$ is an element of area on the manifold which satisfies $\mathrm{d}S = \sqrt{D}\,\mathrm{d}x\,\mathrm{d}y$.

In both one and two dimensions, Equation (10) can become very stiff and it is sometimes necessary to add small regularization terms or internodal pressure terms to prevent the time integration from failing; see Reference [1] for GWMFE and Reference [4] for SGWMFE.

3. SHALLOW WATER EQUATIONS TEST PROBLEM

We test the SGWMFE method using the shallow water equations where a smooth initial 'hump' of stationary water is released at time zero; as the hump subsides under gravity, a wave propagates away and the wave front steepens. The wave is then reflected at the further boundary. In this problem, a simulation should capture the front formation, wave height and front speed.

In one dimension, the shallow water equations (with addition of artificial viscosity ε) for the flow of a fluid with unit density and unit gravitational constant over a flat bottom are

$$u_t + f_x = \varepsilon u_{xx}, \quad v_t + g_x = \varepsilon v_{xx}, \quad \text{with} \quad f = v, \quad g = \left(\frac{v^2}{u} + \frac{u^2}{2}\right) \tag{11}$$

where $0 \leqslant x \leqslant 10$, u is the height of the fluid from the flat bottom and v is the fluid momentum in the x-direction. For a systematic derivation of the shallow water theory, see Reference [6].

The initial conditions are given by $u(x,0) = 0.2 + \exp(-x^2)$, $v(x,0) = 0$. We require $u_x = 0$ and $v = 0$ at both boundaries.

The shallow water equations are extended to two dimensions by adding momentum w in the y-direction:

$$u_t + \nabla \cdot \mathbf{f} = \varepsilon \nabla^2 u, \quad v_t + \nabla \cdot \mathbf{g} = \varepsilon \nabla^2 v, \quad w_t + \nabla \cdot \mathbf{h} = \varepsilon \nabla^2 w \tag{12}$$

with

$$\mathbf{f} = (v, w), \quad \mathbf{g} = (v^2/u + u^2/2, vw/u), \quad \mathbf{h} = (vw/u, w^2/u + u^2/2)$$

For this problem, we use initial conditions $v = w = 0$ and $u(x,0) = 0.2 + \exp(-x^2 - y^2))$ and solve on a square $0 \leqslant x \leqslant 5$, $0 \leqslant y \leqslant 5$ with reflective boundary conditions: $\partial u/\partial n = 0$ on the four sides, $\partial w/\partial n = 0$ and $v = 0$ on $x = 0$ and $x = 5$ and $\partial v/\partial n = 0$ and $w = 0$ on $y = 0$ and $y = 5$. Symmetry allows the solution to be extended to $-5 \leqslant x, y \leqslant 5$ so that the flow is the same as water collapsing under gravity in a box with reflective walls.

4. RESULTS

Solutions for the one-dimensional shallow water equations using GWMFE and SGWMFE have been given in Reference [1], respectively. Solutions for the two-dimensional shallow water equations using GWMFE are found in [7]. In this paper, we show in Figure 1 the nodal trajectories using SGWMFE in an $x - t$ plot for varying initial distribution of nodes and for varying internodal pressures showing how this pressure can help to maintain mesh structure. The surface height using SGWMFE for a one-dimensional problem is shown in perspective view and plan view in Figure 2, illustrating again that gradient weighted methods can capture a very sharp front. For the two-dimensional problem, we show in Figure 3 the development in time of the surface height as the initial hump subsides under gravity and is reflected from boundaries. The evolving triangular mesh is shown in Figure 4. It can be seen that the nodes concentrate in the regions of large change in the solution. In Figure 5, we show the effect of using three different starting meshes. In each case, the meshes adjust very quickly to be qualitatively similar as the solution manifold is tracked.

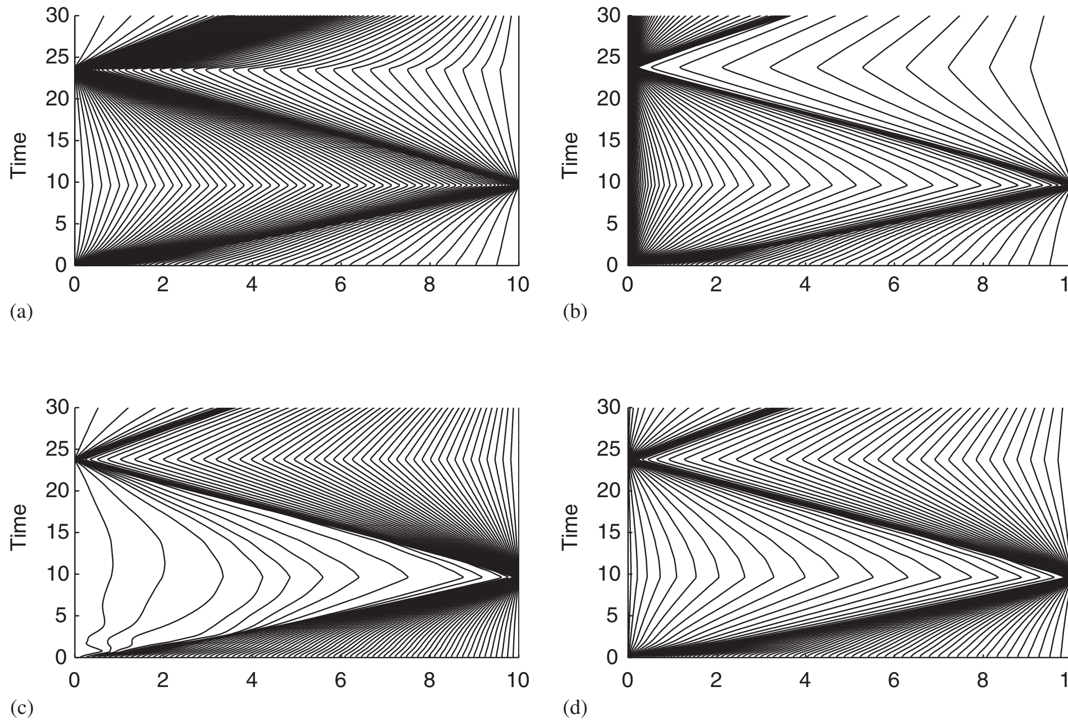

Figure 1. Nodal trajectories, all with $\varepsilon = 0.005$: (a) $x(\tau, 0) = 9.9\tau^4 + 0.1\tau$, $0 \leqslant \tau \leqslant 1$, internodal pressure 10^{-7}; (b) $x(\tau, 0) = 9.9\tau^4 + 0.1\tau$, $0 \leqslant \tau \leqslant 1$, internodal pressure 10^{-11}; (c) $x(\tau, 0) = 10\tau$, $0 \leqslant \tau \leqslant 1$, internodal pressure 10^{-11}; and (d) $x(\tau, 0) = 9.9\tau^2 + 0.1\tau$, $0 \leqslant \tau \leqslant 1$, internodal pressure 10^{-11}. See Reference [1] for discussion of internodal pressure.

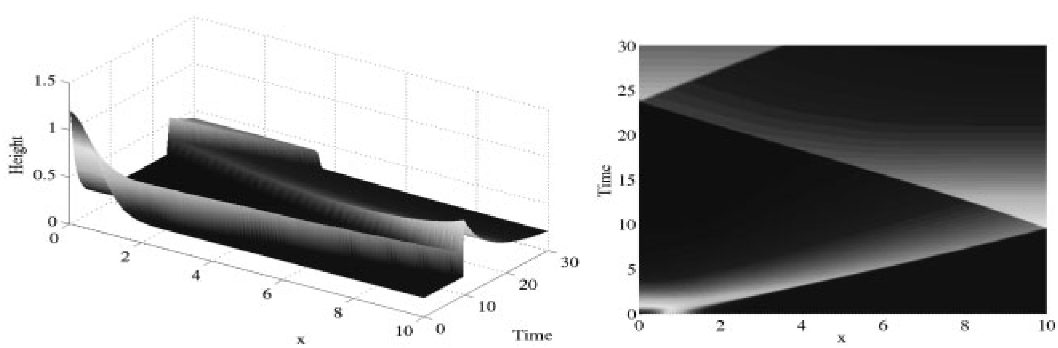

Figure 2. Surface height for one-dimensional problem showing 3D and contour view.

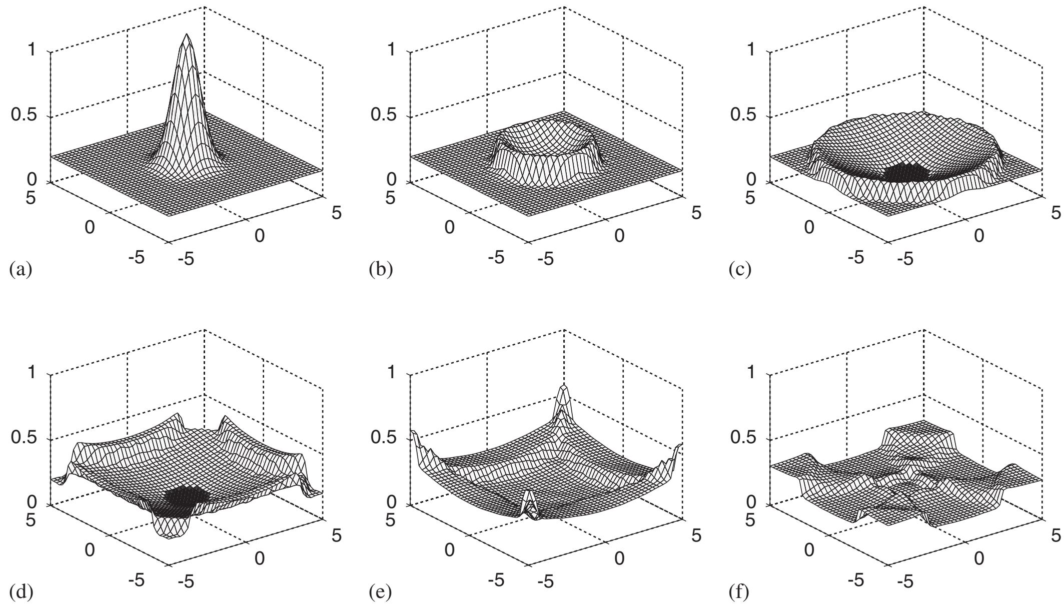

Figure 3. Water depth for 2D problem: (a) $t=0$; (b) $t=2$; (c) $t=5$; (d) $t=7$; (e) $t=9$; and (f) $t=11$.

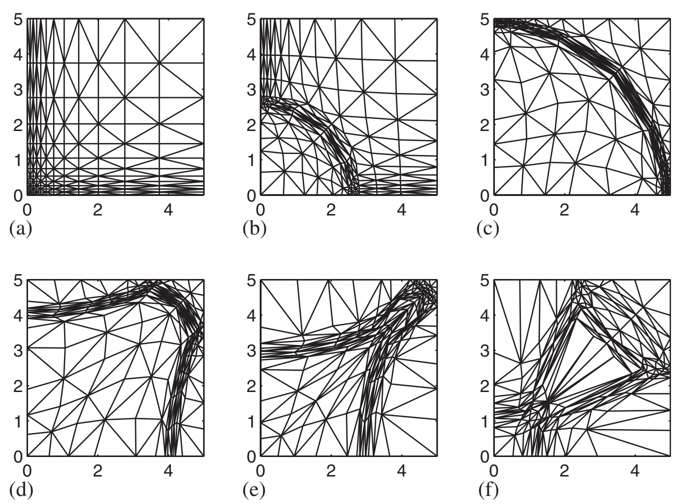

Figure 4. Evolution of mesh for 2D problem: (a) $t=0$; (b) $t=2$; (c) $t=5$; (d) $t=7$; (e) $t=9$; and (f) $t=11$.

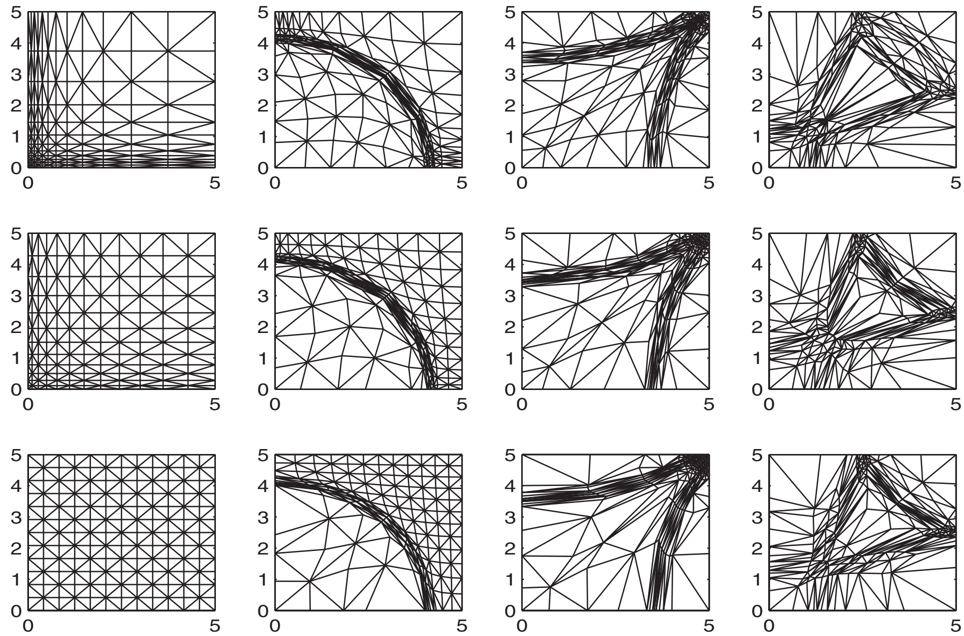

Figure 5. Evolution of different starting meshes; each row gives the mesh at times $t = 0, 4, 8, 12$.

5. CONCLUSIONS

We have implemented the SWGMFE method for systems of PDEs with one and two space dimensions. The results for shallow water equations show that the method is robust and easy to generalize to multiple space dimensions and multiple equations. Examining different starting meshes shows that the nodes move to positions which, as expected, are determined by the evolving manifold much more than the starting mesh.

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